(Fast) Elliptic Solves for Constrained Evolution and BBH Initial Data

Scott H. Hawley* and Richard A. Matzner

with Matthew Anderson, Erin Bonning and Michael Vitalo

University of Texas at Austin

** shawley@physics.utexas.edu*

Abstract: In the course of a numerical evolution of a (3+1) spacetime, numerical errors typically result in deviations from the constraint hypersurface. Experiences with "fully constrained" codes [1] indicate that greater stability can be gained if the constraints are enforced frequently throughout the simulation. One reason for the lack of 3D constrained evolutions performed to date is that the elliptic solves involved in enforcing the constraints can be computationally expensive. Thus we have been motivated to develop a fast elliptic solver based on the O(n) multigrid method [2]. The presence of an excised region in the domain presents a challenge which we have found a novel and simple treatment for, as described in [3]. In that previous work, we described the solution of the Hamiltonian constraint for a single black hole on a flat background. In the present work, we describe the generalization of this to the full set of constraint equations. We detail some of the specifics of our implementation, and give some reliminary results toward the goal of solving for the Kerr-Schild type initial data of Bonning et al. [4].

Physical System

York-Lichnerowitz conformal method: Background $\{\tilde{g}_{ij},\tilde{K}_{ij}\}$, Physical $\{g_{ij},K_{ij}\}$

$$\left(A_{ij} \equiv K_{ij} - \frac{1}{3}\delta_{ij}K^k{}_k\right)$$

$$g_{ij} = \phi^4 \tilde{g}_{ij},$$

$$A^{ij} = \phi^{-10}(\tilde{A}^{ij} + (\tilde{lw})^{ij})$$

$$K = \tilde{K}$$

$$(\tilde{lw})^{ij} \equiv \tilde{\nabla}^i w^j + \tilde{\nabla}^j w^i - \frac{2}{3}\tilde{g}^{ij}\tilde{\nabla}_k w^k$$

$$\tilde{\nabla}^2\phi = (1/8)(\tilde{R}\phi + \frac{2}{3}\tilde{K}^2\phi^5 - \phi^{-7}(\tilde{A}^{ij} + (\tilde{lw})^{ij})(\tilde{A}_{ij} + (\tilde{lw})_{ij}))$$

$$\tilde{\nabla}_j(\tilde{lw})^{ij} = \frac{2}{3}\tilde{g}^{ij}\phi^6\tilde{\nabla}_j K - \tilde{\nabla}_j\tilde{A}^{ij}.$$

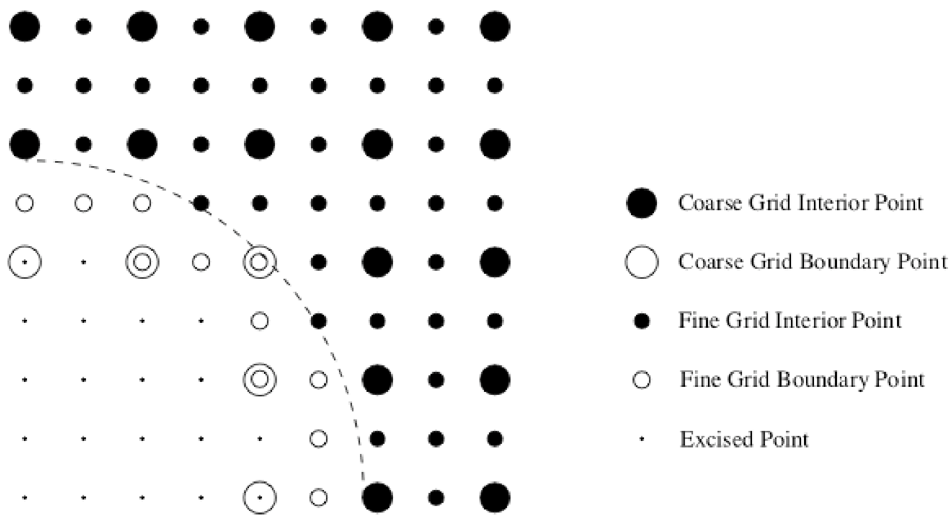
- Dirichlet inner boundary conditions
- Simplified Robin outer boundary conditions
  - $[r(\phi-1)]_{,r} = 0$ implemented normal to faces of cube
  - $w^i = 0$

TEXMEX Code Overview

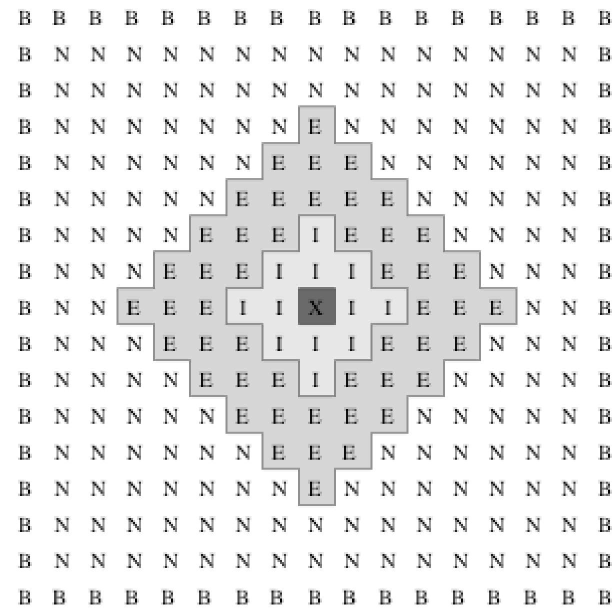
("Texas Multigrid Excision" Code)

- Standalone code (no CACTUS, etc…), designed as a "black box" elliptic solver: input guess/background $\{g_{ij}, K_{ij}\}$, output new $\{g_{ij}, K_{ij}\}$ satisfying constraints
- Vertex-centered
- Similar to other multigrid solvers (e.g., Bruegmann, Pretorius)
  - but with unique inner boundary/restriction scheme, and a few other modifications "under the hood"
- Code for Kerr-Schild "superposition" background by Pedro Marronetti (w/ tweaks by Matt Anderson)
- Heart of constraint solver (residual evaluator) by Matt Anderson
- Newton-Gauss-Seidel smoothing iterations
- "Extra Smoothing Region" (ESR) - solve twice as often near hole
- Simple SOR for coarse-grid solves (switches over to Gauss-Seidel near target tolerance)
- Can run with reflection symmetries (octant, bitant)
- Working on parallelism with student Mike Vitalo
- Coarse grid inner boundary values supplied via direct copy from fine grid points where possible, otherwise via weighted multidirectional extrapolation

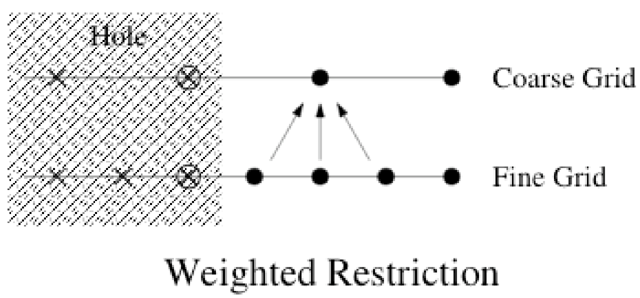
Implementation Details

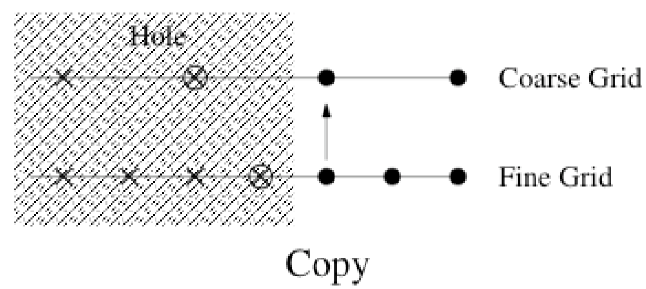


Inner boundary points are those points which are immediately interior to a circle of radius r_{max}. Here we show a fine grid and a coincident coarse grid.



2D schematic of mask function, showing excised points (X), inner boundary points (I), Extra Smoothing Region (E), normal interior points (N), and outer boundary points (B)

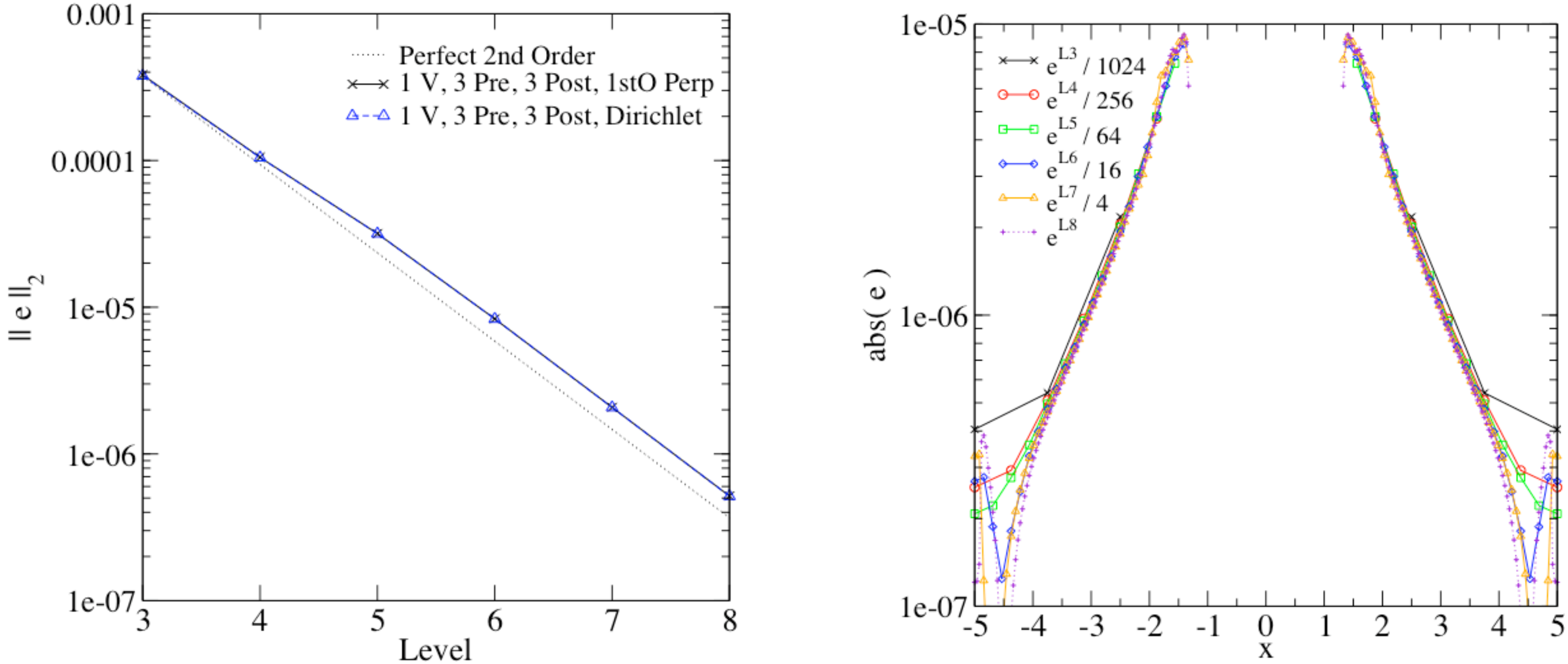




1D schematic of scheme for inner boundary and restriction scheme. Filled circles denote normal interior points, Xs denote excised pionts, and the open circles with Xs through them denote inner boundary points (where the Dirichlet conditions are applied).

Results: Flat Background

Convergence Plots

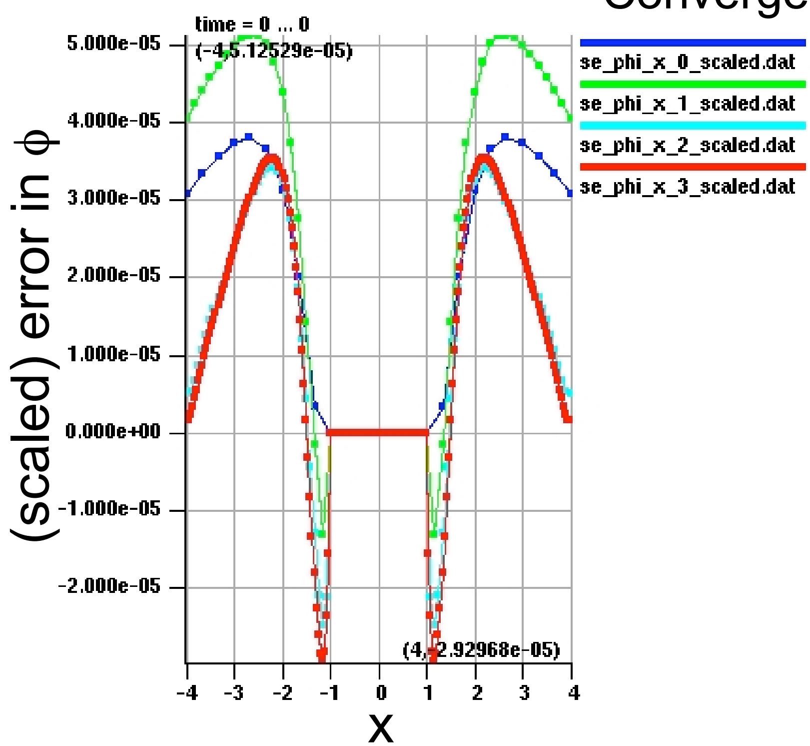


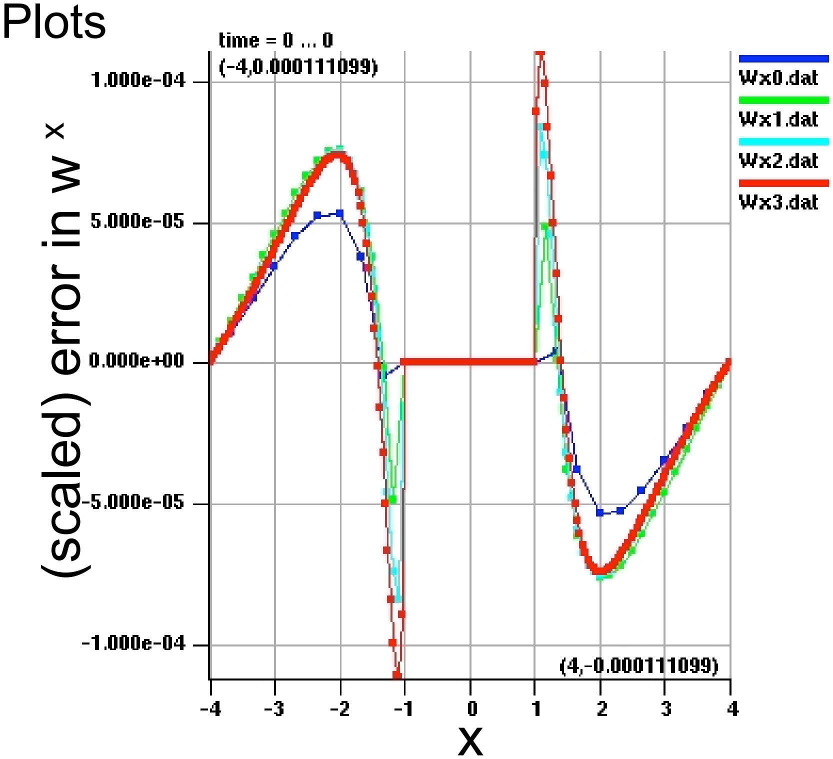
(Grid has 2^{Level}-1 points along each side)

Convergence results for 3D solutions to the Hamiltonian constraint of the form $\phi = 1 + 2\,M/r$, for runs in which r_{mask}=1.29, for a domain [-5,-5,-5] to [5,5,5]. **Left panel**: A logarithmic plot of the L2 norm of the solution error e, showing a comparison between outer boundary conditions. Using the ``first order perpendicular'' (FOP) implementation of the Robin boundary condition, we obtain convergence results which lie on top of those obtained using a Dirichlet outer boundary condition. These results also run parallel to the line for perfect second order behavior. **Right Panel:** A logarithmic plot of the solution error e itself, at the end of each V-cycle in the Full Multigrid Algorithm. Here we have divided coarser grid values by appropriate powers of four in order to make the comparison. Near the outer boundary, the magnitude of the error is roughly second order, however its shape is resolution-dependent. This feature may arise from the use of the FOP condition.

Results: Schwarzschild Background

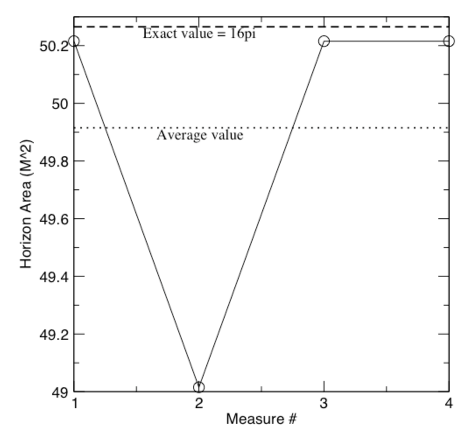
Convergence Plots



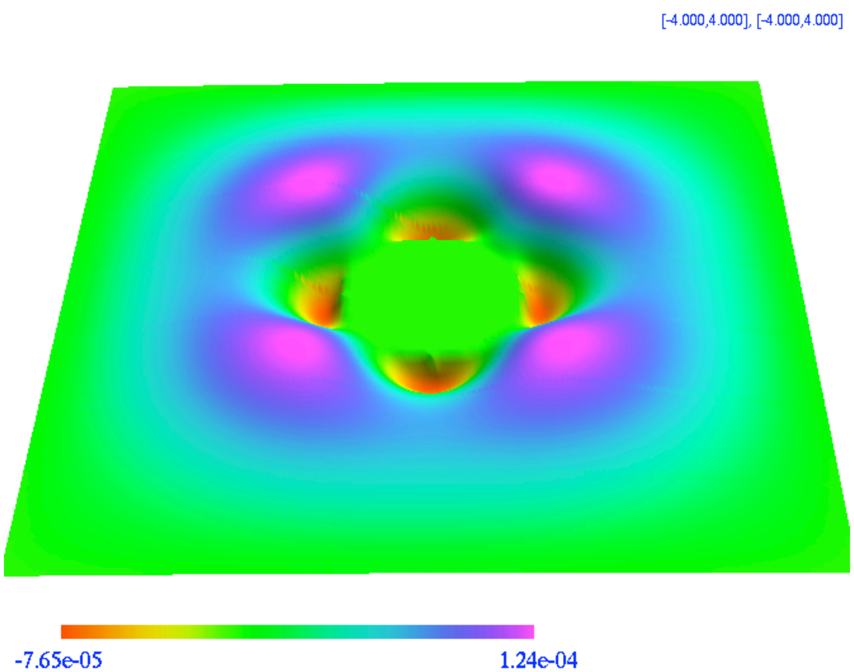


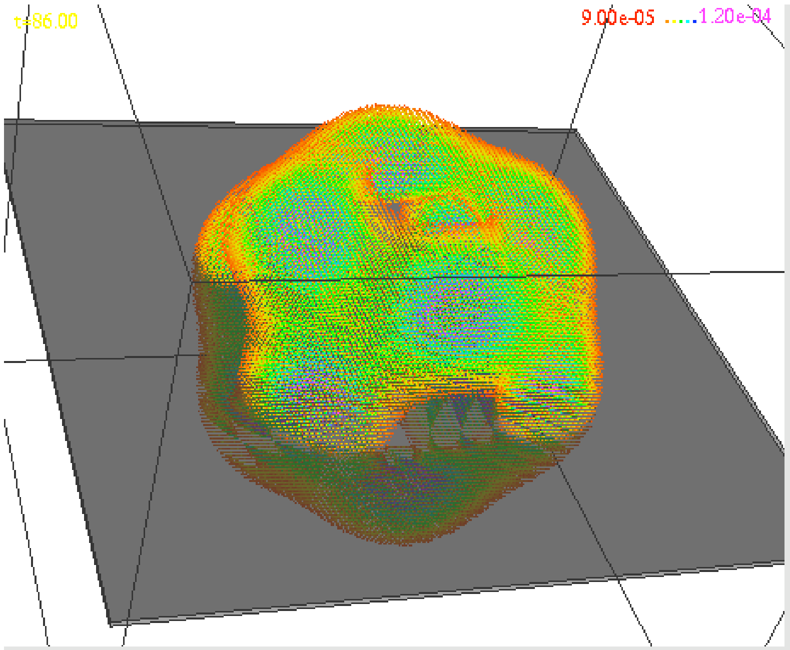
Horizon Area (via Erin Bonning's code)

Final Area of Apparent horizon (measure 1) = 50.215497
Final Area of Apparent horizon (measure 2) = 49.015306
Final Area of Apparent horizon (measure 3) = 50.215497
Final Area of Apparent horizon (measure 4) = 50.215497
Final Area of Apparent horizon (average) = 49.915449
Exact value (16π) = 50.265482



Schwarzschild Background cont'd





Solution error in phi (e = ϕ -1) on finest grid (129^3). We expect a spherically-symmetric solution, however the behavior of the solution along "flat" sides of the excision mask is different from the behavoir along "staircase" (or "diagonal") sides of the excision mask, giving rise to this "lumpy" solution. Increasing the resolution lowers the magnitude of these variations, but not their angular distribution.

Future Work

- Physics:
  - Obtain convergent solutions for Kerr-Schild-type BBH initial data
  - Study BBH binding energy as function of separation & spins
  - Integrate TEXMEX w/ Matt Anderson's constrained evolution code, to more efficiently continue investigation of constrained evolution.
- Computer Code:
  - Timing comparisons to see "how fast"
  - Parallelization currently underway with student Michael Vitalo
  - Add mesh refinement

References

1. e.g., Choptuik M W, Hirschmann E W, Liebling S L, and Pretorius F 2003, *Class. Quant. Grav.* **20**, 1857; Anderson M and Matzner R A 2003 gr-qc/0307055 ; Bonazzola S, Gourgoulhon E, Grandclement P, Novak J, gr-qc/0406020; Schnetter E, Penn State NR Lunch, Feb 13 2004.
2. Brandt A 1977 *Math. of Computation* **31**, 333.
3. Hawley S H and Matzner R A, *Class.Quant.Grav*. **21** (2004) 805.
4. Bonning E, Marronetti P, Neilsen D and Matzner R A 2003 *Phys. Rev*.**D68**, 044019.