PHY2010: Simple Harmonic Motion

Simple Harmonic Motion (SHM) is an indispensable paradigm for understanding a wide variety of physical phenomena, including much of acoustics.

The canonical system demonstrating SHM is a mass m connected to a spring with spring constant k:

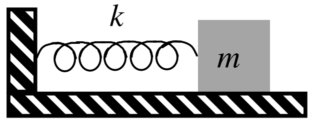

Figure 1: The canonical simple harmonic oscillator, a mass on a spring.

The force due to the spring is described by *Hooke's Law*, $F = -kx$, where x is the displacement from equilibrium. Note that via Newton's Second Law, $F = ma$, one can relate the acceleration a to the displacement x,

$$-kx = ma$$

$$a + (k/m)x = 0.$$

If we consider that a is "the time rate of change of the time rate of change" of x, then the function $x(t)$ which satisfies the above equation is a sine or a cosine function, such as:

$$x(t) = A\cos(\omega_0 t + \phi), \tag{1}$$

where A is a (constant) amplitude, and the value ω_0 is the natural "angular frequency" of the oscillator, in radians per second. The variable ϕ in the equation for $x(t)$ is a constant angle or "phase," describing what the oscillator does at time $t = 0$. (Note that for $\phi = -\pi/2$, the function $x(t)$ can be written $x(t) = A\sin(\omega_0 t)$.)

The value of ω_0 is the square root of k/m that appeared in Newton's Second Law,

$$\omega_0 = \sqrt{\frac{k}{m}} \text{ (natural angular freq.)},$$

and is related to the natural frequency f_0 in Hertz via $\omega_0 = 2\pi f_0$:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ (natural freq. in Hz)}.$$

Damped Oscillations

The presence of *friction* in the system causes the *dissipation of energy* from the oscillator. The oscillator must do *work* against friction as the mass moves back and forth, and this causes energy to leave the oscillator as it gets converted into *heat*. This leads to an *exponential* decay of the amplitude as a function of time, shown in Figure 2 below. This is also known as the *impulse response* or *transient response* of the oscillator: if you "hit" it briefly, it will oscillate at its natural frequency ω_0 and damp out.

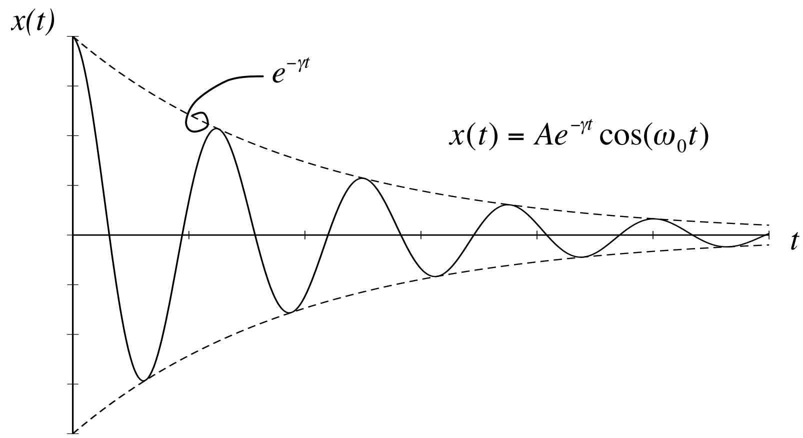

Figure 2: Impulse or "transient" response of a damped oscillator.

Driven, Damped Oscillations

If you drive the system with a force that oscillates at (some other) frequency f, then the system will oscillate at the driving frequency. The presence of the driving force means that energy can be continually added to the system (while it is being dissipated by the damping) and we will have a *steady state* oscillation (which doesn't decay), in which the amplitude A will be a function of the driving frequency, *i.e.* $A(f)$, as shown in Figure 3.

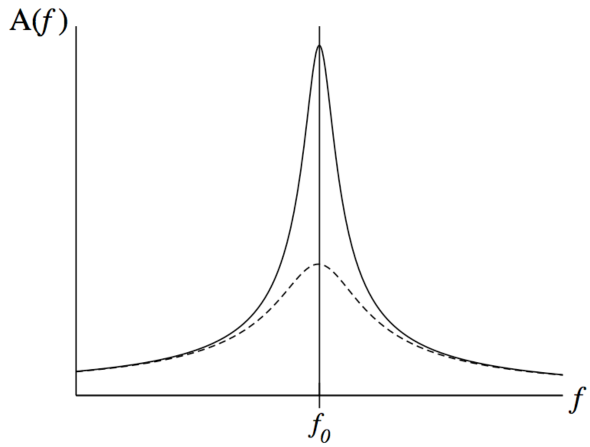

Figure 3: Steady-state response of a driven damped oscillator. The solid line shows little damping, the dashed line shows much damping.

The maximum amplitude occurs when $f = f_0$, which is known as *resonance*. In other words, *resonance occurs when the frequency of the driving force is equal to the natural oscillation frequency of the system.*

The Q Factor

For any damped oscillator, the rate at which energy is dissipated from the system is related to the amount of damping. One way of describing the amount of damping quantitatively is via the (dimensionless) *Q factor* or "quality" factor, which is equal to the energy stored in the oscillator a given time divided by the energy lost in one cycle (times a factor of 2π which we won't worry about here.) Thus a large Q denotes small damping, and vice versa.

Another way of defining the Q factor, which is equivalent to the first way when the damping is weak enough (and in acoustics this is typically the case), is by the "width" of the "resonance" peak in the steady-state response graph. For this, we actually graph the *energy* E in the oscillator as a function of the driving frequency f, and then define Δf to be the "full width at half max" of the E vs. f graph, as shown in Figure 4.

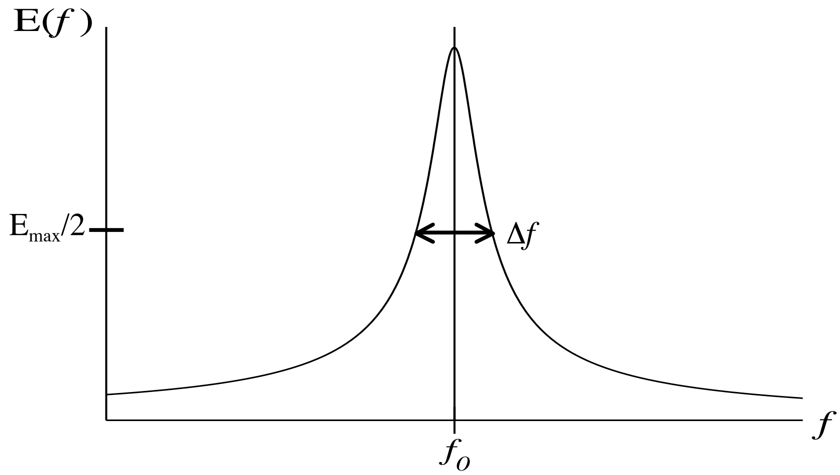

Figure 4: Defining Δf, the full width at half the maximum energy.

The Q value is then given by

$$Q = \frac{f_0}{\Delta f}.$$