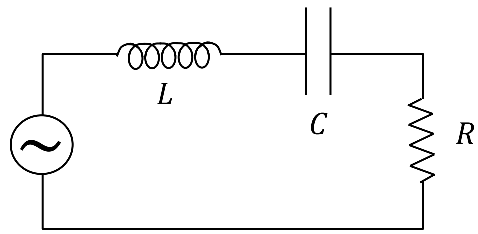
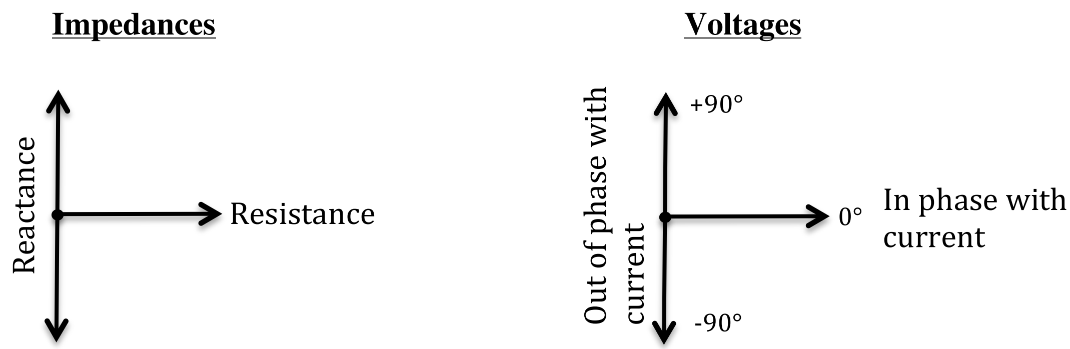
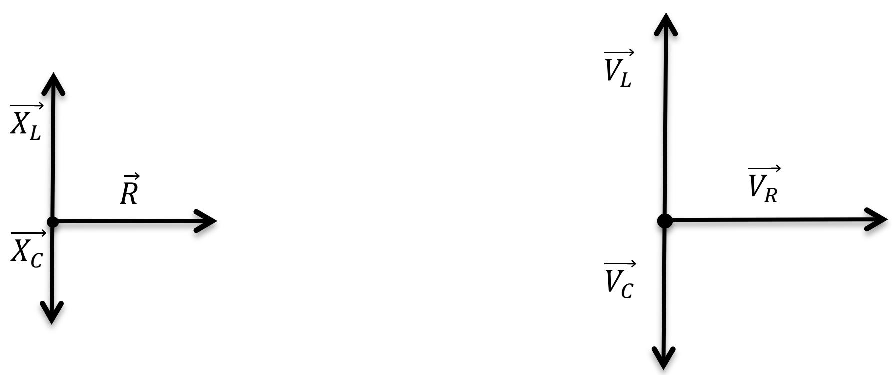
Series RLC Filters and Geometry
or
The Mighty Impedance Diagram!





The impedance diagram on the left and the voltage diagram on the right will always be geometrically *similar*, meaning that all angles will be the same, and the relative lengths of vectors will be the same. In the following, we will draw the voltage diagram at a scale larger than that of the impedance diagram, just to emphasize this "similarity" property.



Where, for a given frequency f, the lengths of the reactance vectors are $X_L = 2\pi f L$ and $X_C = 1/(2\pi f C)$. (Note that we have drawn these diagrams assuming the inductor has a larger contribution than the capcitor, but the lengths may be different in general.)
Because of similarity, relative lengths are the same. So for example,

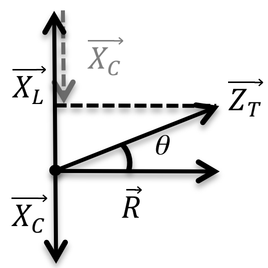
$$\frac{R}{X_L} = \frac{V_R}{V_L}$$

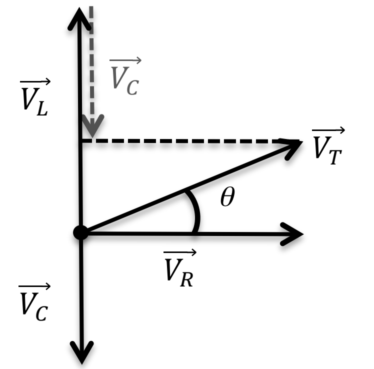
If we want to include the total impedance Z_T and total voltage V_T, these are simply the vector sums of their corresponding parts, i.e.

$$\overrightarrow{Z_T} = \vec{R} + \overrightarrow{X_C} + \overrightarrow{X_L}$$

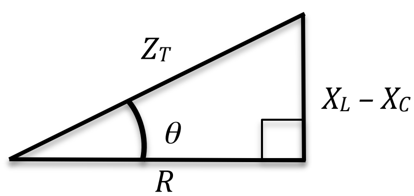
$$\overrightarrow{V_T} = \overrightarrow{V_R} + \overrightarrow{V_C} + \overrightarrow{V_L}$$

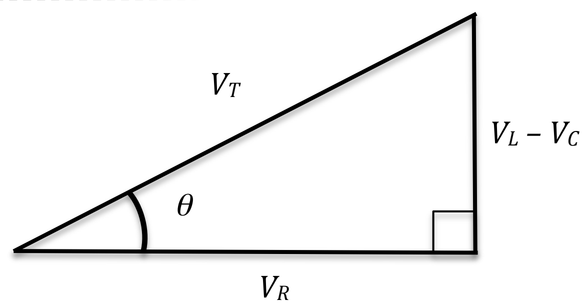
as shown in the diagrams:





Where similarity tells us that θ is the same in both diagrams. Let us reduce the above diagrams into mere triangles, which we will "zoom in" on:





where vertical bars denote absolute value. (Note that we've drawn the triangles "upward," but they could just as easily be drawn "downward" for $X_C > X_L$.) Using these triangles, let's write out a few trigonometric identities:

$$Z_T = \sqrt{R^2 + (X_L - X_C)^2} \qquad V_T = \sqrt{{V_R}^2 + (V_L - V_C)^2}$$

$$\cos\theta = \frac{R}{Z_T} = \frac{V_R}{V_T}\,^{*}$$

$$\sin\theta = \frac{X_L - X_C}{Z_T} = \frac{V_L - V_C}{V_T}$$

$$\tan\theta = \frac{X_L - X_C}{R} = \frac{V_L - V_C}{V_R}$$

The angle θ will be positive or negative depending on whether the total voltage is *leading* (+) or *lagging* (-) the current.

* Note that this is just a "voltage divider" formula, with R_T replaced by Z_T.

<u>Example</u>: Bandpass Filter

Say we measure the voltage across the resistor, i.e. we attach our "output" in series with L and C. The voltage gain G (or "transfer function") of a filter is given by $G = V_{out}/V_{in}$, and in this case $V_{out} = V_R$ and $V_{in} = V_T$. So our G is simply

$$G = \frac{V_R}{V_T}$$

Using the impedance side of the diagrams above: $V_R/V_T = R/Z_T$. Thus G is

$$G = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}}$$

Let's pretty this up: Multiply both the denominator and the numerator by $1/R$ to find

$$G = \left[1 + \left(\frac{X_C}{R} - \frac{X_L}{R}\right)^2\right]^{-1/2} = \left[1 + \left(\frac{1}{2\pi f CR} - \frac{2\pi f L}{R}\right)^2\right]^{-1/2}$$

Then we can group "variables that are not f" into two new variables f_1 and f_2 such that

$$f_1 = \frac{1}{2\pi RC}$$

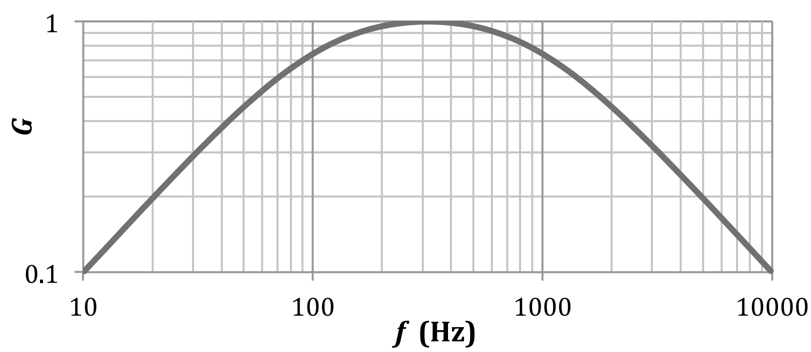
and

$$f_2 = \frac{R}{2\pi L}$$

With these substitutions, we arrive at

$$G = \left[1 + \left(\frac{f_1}{f} - \frac{f}{f_2}\right)^2\right]^{-1/2} \qquad (1)$$

To see this in action, let's choose some values: $R = 8\ \Omega$, $f_1 = 100$ Hz and $f_2 = 1000$ Hz. Our choice of f_1 means that $C = 199\ \mu$F and our choice of f_2 means that $L = 1.27$ mH. (Check: derive these values yourself!) The graph below shows a Bode (log-log) plot of the gain vs. frequency†:



Note that if $f_1 \to 0$ ($C \to \infty$), we have purely a high pass filter, and if $f_2 \to \infty$ ($L \to 0$) we have purely a low pass filter. Thus Eq. (1) includes low-, high-, and band-pass filters.

This result is only *one example* of the many things *you* can figure out using an impedance diagram and trigonometry!

Exercise: For the values in the above example, find f for which the total impedance is purely resistive, *i.e.*, $Z_T = R$. What is the gain at this frequency? Answers below†.

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† What is often of interest may not be the voltage *per se* but rather the *power*, the gain for which will be the *square* of G. Thus the slope of the "sides" for the power graph will be +/-2, not +/-1 as above.

†Answers: 316 Hz, G =1